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On Infinite Bernoulli Matrices

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Abstract. In this work, we study some properties of infinite Bernoulli matrices. Further, we investigate relations between infinite Bernoulli matrices and some infinite matrices such as Fibonacci, Pascal and special matrices.

INTRODUCTION

The generalized Bernoulli polynomials $B_k^{(a)}(x)$ can be defined by the generating formula

$$\frac{t^a e^{xt}}{(e^t-1)^a} = \sum_{k=0}^{\infty} \frac{t^k}{k!} B_k^{(a)}(x), \quad |t| < 2\pi, \tag{1}$$

where a is a real or complex parameter. $B_k^{(a)}(x)$ is a polynomial in x of degree k.

Using the equation (1), we can obtain that

$$B_n(x) = \sum_{k=0}^{n} \binom{n}{k} B_{n-k} x^k = \sum_{k=0}^{n} \binom{n}{k} B_k x^{n-k}. \tag{2}$$

where B_k is a Bernoulli number. Their most important other properties as follows:

$$\begin{aligned} &B'_n(x) = nB_{n-1}(x) \\ &B_n(x+1) - B_n(x) = nx^{n-1} \\ &B_n(1-x) = (-1)^n B_n(x). \end{aligned}$$

If choose $a = 1$, then, $B_k^{(1)}(x) = B_k(x)$ and also choose $x = 0$, then $B_k(x) = B_k$. Therefore, $B_k^{(0)}(x) = x^k$ and $B_0^{(a)}(x) = 1$. Further, $B_k^{(1)}(x) = B_k(x)$ and $B_k^{(1)}(0) = B_k$ are Bernoulli polynomials and Bernoulli numbers, respectively. That is, the values of $B_k(x)$ at $x = 0$ are called Bernoulli numbers.

Bernoulli numbers by the corresponding equation

$$\frac{t}{e^t - 1} = \sum_{k=0}^{\infty} B_k \frac{t^k}{k!}.$$

Bernoulli numbers are usually defined by the recursive relations:

$$B_0 = 1, \quad \sum_{j=0}^{n} \binom{n+1}{j} B_j, \quad n > 0,$$

in particular for $k = 0, 1, 2, \cdots$, the first few Bernoulli numbers are $B_0 = 1$, $B_1 = -1/2$, $B_2 = 1/6$, $B_4 = -1/30$, $B_6 = 1/42$, $B_8 = -1/30$, $B_{10} = 5/66$, $\cdots$ with $B_{2k+1} = 0$ for each $k \geq 1$.

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TABLE 1. The first six Bernoulli polynomials

$B_0(x)$	1
$B_1(x)$	$x - 1/2$
$B_2(x)$	$x^2 - x + 1/6$
$B_3(x)$	$x^3 - 3/2x^2 + 1/2x$
$B_4(x)$	$x^4 - 2x^3 + x^2 - 1/30$
$B_5(x)$	$x^5 - 5/2x^4 + 5/3x^3 - 1/6x$

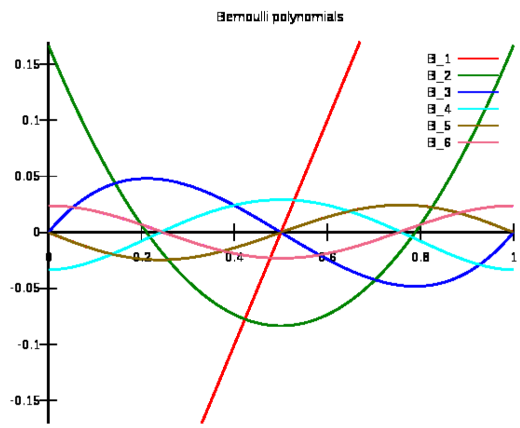

FIGURE 1. The behavior of the first six of Bernoulli polynomials.

SOME CERTAIN PROPERTIES

Theorem 1 *The operators $X : \omega \to \omega$ are linear over $\mathbb{C}$, where $X \in \{B, B^{-1}\}$ and ω is the set of all real and complex valued sequences.*

Theorem 2 *Let I_B an identity operator in ω. Then, $B \circ B^{-1} = B^{-1} \circ B = I_B$.*

Definition 1 *The infinite matrix A is called regular if $\lim Ax = \lim x$ for every $x \in c$.*

In theory of infinite matrices, finding an inverse of an infinite matrix is an extremely important one. Finding the inverse of the infinite matrix allows many results to be formulated and proven.

Lemma 1 *[1] A lower triangular matrix $\mathcal{A} = (a_{nk})$ is invertible if and only if $a_{nn} \neq 0$, for all $n \in \mathbb{N}$.*

Corollary 1 *For all $k, n \in \mathbb{N}$, the Bernoulli matrix $\mathcal{B} = (b_{nk})$ is invertible if and only if $b_{nn} \neq 0$.*

Example 1 *The infinite matrix $\mathcal{D} = (d_{nk})$ of the inverse of infinite Bernoulli matrix $\mathcal{B} = (b_{nk})$ defined as follows:*

$$d_{nk} = \begin{cases} \frac{1}{n-k+1}\begin{pmatrix} n \\ k \end{pmatrix} & , \quad (n \geq k) \\ 0 & , \quad (other) \end{cases}$$

for all $k, n \in \mathbb{N}$.

Proof.

$$\mathcal{B}.\mathcal{D} = \begin{bmatrix} 1 & 0 & 0 & 0 & 0 & \cdots \\ -1/2 & 1 & 0 & 0 & 0 & \cdots \\ 1/6 & -1 & 1 & 0 & 0 & \cdots \\ 0 & 1/2 & -3/2 & 1 & 0 & \cdots \\ -1/30 & 0 & 1 & -2 & 1 & \cdots \\ \vdots & \vdots & \vdots & \vdots & \vdots & \ddots \end{bmatrix} \times \begin{bmatrix} 1 & 0 & 0 & 0 & 0 & \cdots \\ 1/2 & 1 & 0 & 0 & 0 & \cdots \\ 1/3 & 1 & 1 & 0 & 0 & \cdots \\ 1/4 & 1 & 3/2 & 1 & 0 & \cdots \\ 1/5 & 1 & 2 & 2 & 1 & \cdots \\ \vdots & \vdots & \vdots & \vdots & \vdots & \ddots \end{bmatrix}$$

$$= \begin{bmatrix} 1 & 0 & 0 & 0 & 0 & \cdots \\ 0 & 1 & 0 & 0 & 0 & \cdots \\ 0 & 0 & 1 & 0 & 0 & \cdots \\ 0 & 0 & 0 & 1 & 0 & \cdots \\ 0 & 0 & 0 & 0 & 1 & \cdots \\ \vdots & \vdots & \vdots & \vdots & \vdots & \ddots \end{bmatrix} = I = \mathcal{D}.\mathcal{B}$$

■ The infinite Pascal matrix $\mathcal{P}(x) = (p_{nk})$ is defined by

$$p_{nk} = \begin{cases} \begin{pmatrix} n \\ k \end{pmatrix} x^{n-k} & , \quad (n \geq k) \\ 0 & , \quad (other) \end{cases}$$

for all $k, n \in \mathbb{N}$ and $\mathcal{P}^{-1}(x) = \mathcal{P}(-x)$ [2].

Zhang and Wang[3] given an example related to the product of $(n+1)\times(n+1)$ Bernoulli polynomial matrix and $(n+1)\times(n+1)$ Pascal matrix. The following example is the generalization of the example of Zhang and Wang[3] with infinite matrices.

Example 2 $\quad \mathcal{P}(x).\mathcal{B} = \mathcal{B}(x)$

Proof.

$$\mathcal{P}(x).\mathcal{B} = \begin{bmatrix} 1 & 0 & 0 & 0 & 0 & \cdots \\ x & 1 & 0 & 0 & 0 & \cdots \\ x^2 & 2x & 1 & 0 & 0 & \cdots \\ x^3 & 3x^2 & 3x & 1 & 0 & \cdots \\ x^4 & 4x^3 & 6x^2 & 4x & 1 & \cdots \\ \vdots & \vdots & \vdots & \vdots & \vdots & \ddots \end{bmatrix} \times \begin{bmatrix} 1 & 0 & 0 & 0 & 0 & \cdots \\ -1/2 & 1 & 0 & 0 & 0 & \cdots \\ 1/6 & -1 & 1 & 0 & 0 & \cdots \\ 0 & 1/2 & -3/2 & 1 & 0 & \cdots \\ -1/30 & 0 & 1 & -2 & 1 & \cdots \\ \vdots & \vdots & \vdots & \vdots & \vdots & \ddots \end{bmatrix}$$

$$\begin{bmatrix} 1 & 0 & 0 & 0 & 0 & \cdots \\ x-\frac{1}{2} & 1 & 0 & 0 & 0 & \cdots \\ x^2-x+\frac{1}{6} & 2x-1 & 1 & 0 & 0 & \cdots \\ x^3-\frac{3}{2}x^2+\frac{1}{2}x & 3x^2-3x+1/2 & 3x-\frac{3}{2} & 1 & 0 & \cdots \\ x^4-2x^3+x^2-\frac{1}{30} & 4x^3-6x^2+2x & 6x^2-6x+1 & 4x-2 & 1 & \cdots \\ \vdots & \vdots & \vdots & \vdots & \vdots & \ddots \end{bmatrix} = \mathcal{B}(x)$$

■

Example 3 $\quad \mathcal{B}^{-1}(x) = \mathcal{D}.\mathcal{P}^{-1}(x)$.

Proof.

$$\mathcal{B}^{-1}(x)=\begin{bmatrix} 1 & 0 & 0 & 0 & 0 & \cdots \\ x-\frac{1}{2} & 1 & 0 & 0 & 0 & \cdots \\ x^2-x+\frac{1}{6} & 2x-1 & 1 & 0 & 0 & \cdots \\ x^3-\frac{3}{2}x^2+\frac{1}{2}x & 3x^2-3x+1/2 & 3x-\frac{3}{2} & 1 & 0 & \cdots \\ x^4-2x^3+x^2-\frac{1}{30} & 4x^3-6x^2+2x & 6x^2-6x+1 & 4x-2 & 1 & \cdots \\ \vdots & \vdots & \vdots & \vdots & \vdots & \ddots \end{bmatrix}^{-1}$$

$$=\begin{bmatrix} 1 & 0 & 0 & 0 & 0 & \cdots \\ -x+\frac{1}{2} & 1 & 0 & 0 & 0 & \cdots \\ x^2-x+\frac{1}{3} & -2x+1 & 1 & 0 & 0 & \cdots \\ -x^3+\frac{3}{2}x^2-x+\frac{1}{4} & 3x^2-3x+1 & -3x+\frac{3}{2} & 1 & 0 & \cdots \\ x^4-2x^3+2x^2-x+\frac{1}{5} & -4x^3+6x^2-4x+1 & 6x^2-6x+2 & -4x+2 & 1 & \cdots \\ \vdots & \vdots & \vdots & \vdots & \vdots & \ddots \end{bmatrix}$$

$$\begin{bmatrix} 1 & 0 & 0 & 0 & 0 & \cdots \\ \frac{1}{2} & 1 & 0 & 0 & 0 & \cdots \\ \frac{1}{3} & 1 & 1 & 0 & 0 & \cdots \\ \frac{1}{4} & 1 & \frac{3}{2} & 1 & 0 & \cdots \\ \frac{1}{5} & 1 & 2 & 2 & 1 & \cdots \\ \vdots & \vdots & \vdots & \vdots & \vdots & \ddots \end{bmatrix} \times \begin{bmatrix} 1 & 0 & 0 & 0 & 0 & \cdots \\ -x & 1 & 0 & 0 & 0 & \cdots \\ x^2 & -2x & 1 & 0 & 0 & \cdots \\ -x^3 & 3x^2 & -3x & 1 & 0 & \cdots \\ x^4 & -4x^3 & 6x^2 & -4x & 1 & \cdots \\ \vdots & \vdots & \vdots & \vdots & \vdots & \ddots \end{bmatrix} = \mathcal{D}.\mathcal{P}^{-1}(x)$$

■

The infinite Fibonacci matrix $F=(f_{nk})$

$$f_{nk}=\begin{cases} F_{n-k+1} & , \quad n-k+1\geq 0 \\ 0 & , \quad n-k+1<0 \end{cases}$$

for all $k,n\in\mathbb{N}$. The inverse of Fibonacci matrix $F^{-1}=(f_{nk}^{-1})$ is defined by $f_{nk}^{-1}=1$ for $n=k$, $f_{nk}^{-1}=-1$ for $n=k+1$ or $n=k+2$ and $f_{nk}^{-1}=0$ for otherwise.

Now, we give an infinite polynomials matrix $T(x)=(t_{nk}(x))$ is defined by

$$t_{nk}=\begin{cases} \binom{n}{k}B_{n-k}(x)-\binom{n-1}{k}B_{n-k-1}(x)-\binom{n-2}{k}B_{n-k-2}(x) & , \quad 0\leq k\leq n \\ 0 & , \quad k>n \end{cases}$$

for all $k,n\in\mathbb{N}$.

Example 4 $\quad \mathcal{B}(x)=FT(x)$

Proof.

$$F.T(x)=\begin{bmatrix} 1 & 0 & 0 & 0 & \cdots \\ 1 & 1 & 0 & 0 & \cdots \\ 2 & 1 & 1 & 0 & \cdots \\ 3 & 2 & 1 & 1 & \cdots \\ \vdots & \vdots & \vdots & \vdots & \ddots \end{bmatrix} \times \begin{bmatrix} 1 & 0 & 0 & 0 & \cdots \\ x-3/2 & 1 & 0 & 0 & \cdots \\ x^2-2x-1/3 & 2x-2 & 1 & 0 & \cdots \\ x^3-\frac{5x^2}{2}+\frac{x}{2}+\frac{1}{3} & 3x^2-5x+\frac{1}{2} & 3x-\frac{5}{2} & 1 & \cdots \\ \vdots & \vdots & \vdots & \vdots & \ddots \end{bmatrix}$$

$$\begin{bmatrix} 1 & 0 & 0 & 0 & \cdots \\ x-\frac{1}{2} & 1 & 0 & 0 & \cdots \\ x^2-x+\frac{1}{6} & 2x-1 & 1 & 0 & \cdots \\ x^3-\frac{3}{2}x^2+\frac{1}{2}x & 3x^2-3x+1/2 & 3x-\frac{3}{2} & 1 & \cdots \\ \vdots & \vdots & \vdots & \vdots & \ddots \end{bmatrix} = \mathcal{B}(x)$$

■

Now, we give an infinite polynomials matrix $Q(x) = (q_{nk}(x))$ is defined by

$$q_{nk} = \begin{cases} \binom{n}{k} B_{n-k}(x) - \binom{n}{k+1} B_{n-k-1}(x) - \binom{n}{k+2} B_{n-k-2}(x) & , \quad 0 \le k \le n \\ 0 & , \quad k > n \end{cases}$$

for all $k, n \in \mathbb{N}$.

Example 5 $\mathcal{B}(x) = Q(x)F$.

Proof.

$$Q(x).F = \begin{bmatrix} 1 & 0 & 0 & 0 & \cdots \\ x-3/2 & 1 & 0 & 0 & \cdots \\ x^2-3x+1/6 & 2x-2 & 1 & 0 & \cdots \\ x^3-\frac{9x^2}{2}+\frac{x}{2}+1 & 3x^2-6x+1 & 3x-\frac{5}{2} & 1 & \cdots \\ \vdots & \vdots & \vdots & \vdots & \ddots \end{bmatrix} \times \begin{bmatrix} 1 & 0 & 0 & 0 & \cdots \\ 1 & 1 & 0 & 0 & \cdots \\ 2 & 1 & 1 & 0 & \cdots \\ 3 & 2 & 1 & 1 & \cdots \\ \vdots & \vdots & \vdots & \vdots & \ddots \end{bmatrix}$$

$$\begin{bmatrix} 1 & 0 & 0 & 0 & \cdots \\ x-\frac{1}{2} & 1 & 0 & 0 & \cdots \\ x^2-x+\frac{1}{6} & 2x-1 & 1 & 0 & \cdots \\ x^3-\frac{3}{2}x^2+\frac{1}{2}x & 3x^2-3x+1/2 & 3x-\frac{3}{2} & 1 & \cdots \\ x^4-2x^3+x^2-\frac{1}{30} & 4x^3-6x^2+2x & 6x^2-6x+1 & 4x-2 & \cdots \\ \vdots & \vdots & \vdots & \vdots & \ddots \end{bmatrix} = \mathcal{B}(x)$$

■ From Examples 4, 5 and[3], we can give the following corollaries: Let D is an inverse of the Bernoulli matrix, $P^{-1}(x)$ is an inverse of the Pascal matrix and F is an Fibonacci matrix. Then,

Corollary 2 *The following conditions are hold:*
i. $T^{-1}(x) = D.P^{-1}(x).F$
ii. $T^{-1} = D.F$

Corollary 3 *The following conditions are hold:*
i. $Q^{-1}(x) = F.P^{-1}(x).D$
ii. $Q^{-1} = F.D$

Theorem 3 *For* $n \in \mathbb{N}$,

$$\mathcal{B}_n(x+y) = \mathcal{B}(x).\mathcal{B}(y) = \mathcal{B}(y).\mathcal{B}(x)$$

If we take $\alpha = 1$ and $\beta = 1$ in Theorem 2.1 in [3], this theorem can be proved in a similar way.

Corollary 4 *For all* $k, n \in \mathbb{N}$, *we have*

$$\mathcal{B}_n(x+1) = \sum_{k=0}^{n} \binom{n}{k} \mathcal{B}_k(x).$$

CONCLUSION

In this paper, infinite Bernoulli polynomials matrix $\mathcal{B}(x)$ and Bernoulli matrix $\mathcal{B}$ are investigated. Linearity as an operator and invertibility of the matrix $\mathcal{B}$ are showed. The examples of the matrices $\mathcal{B}(x)$ and $\mathcal{B}$ are given.

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