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Almost Convergence and Generalized Weighted Mean

Murat Kirişçi

Department of Mathematical Education, Hasan Ali Yücel Education Faculty, Istanbul University, Vefa, 34470, Istanbul, Turkey

Abstract. In this paper, we investigate some new sequence spaces which naturally emerge from the concepts of almost convergence and generalized weighted mean. The object of this paper is to introduce to the new sequence spaces obtained as the matrix domain of generalized weighted mean in the spaces of almost null and almost convergent sequences. Furthermore, the beta and gamma dual spaces of the new spaces are determined and some classes of matrix transformations are characterized.

Keywords: Almost convergence, Matrix domain, Sequence spaces, Generalized weighted mean, Beta- and gamma-duals, Matrix transformations
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INTRODUCTION

By a *sequence space*, we understand a linear subspace of the space $\mathbb{C}^{\mathbb{N}}$ of all complex sequences which contains ϕ, the set of all finitely non-zero sequences, where $\mathbb{C}$ denotes the complex field and $\mathbb{N} = 0,1,2,\ldots$. We write ℓ_∞, c and c_0 for the classical spaces of all bounded, convergent and null sequences, respectively. Also by bs, cs, ℓ_1 and ℓ_p, we denote the space of all bounded, convergent, absolutely and p-absolutely convergent series, respectively.

Let λ and μ be two sequence spaces and $A = (a_{nk})$ be an infinite matrix of real or complex numbers a_{nk}, where $k,n \in \mathbb{N}$. Then, we say that A defines a matrix mapping from λ into μ, and we denote it by writing $A : \lambda \to \mu$ if for every sequence $x = (x_k) \in \lambda$. The sequence $Ax = \{(Ax)_n\}$, the A-transform of x, is in μ; where

$$(Ax)_n = \sum_k a_{nk}x_k, \quad (n \in \mathbb{N}). \tag{1}$$

By $(\lambda : \mu)$, we denote the class of all matrices A such that $A : \lambda \to \mu$. Thus, $A \in (\lambda : \mu)$ if and only if the series on the right side of (1) converges for each $n \in \mathbb{N}$ and each $x \in \lambda$ and we have $Ax = \{(Ax)_n\}_{n\in\mathbb{N}} \in \mu$ for all $x \in \lambda$. A sequence x is said to be A-summable to α if Ax converges to α which is called the A-limit of x. Also by $(\lambda : \mu; p)$, we denote the subset of $(\lambda : \mu)$ for which limits or sums are preserved whenever there is a limit or sum on the spaces λ and μ. The matrix domain λ_A of an infinite matrix A in a sequence space λ is defined by

$$\lambda_A = \{x = (x_k) \in \omega : Ax \in \lambda\} \tag{2}$$

which is a sequence space. If A is triangle, then one can easily observe that the sequence spaces λ_A and λ are linearly isomorphic, i.e., $\lambda_A \cong \lambda$. We write U for the set of all sequences $u = (u_k)$ such that $(u_k) \neq 0$ for all $k \in \mathbb{N}$. For $u \in U$, let $1/u = (1/u_k)$. Let $z,u,v \in U$ and define the *generalized weighted mean* or *factorable matrix* $G(u,v) = (g_{nk})$ by

$$g_{nk} = \begin{cases} u_n v_k & , \quad (0 \le k \le n) \\ 0 & , \quad (k > n) \end{cases}$$

for all $k,n \in \mathbb{N}$; where u_n depends only on n and v_k only on k.

The main purpose of present paper is to introduce the sequence spaces $f_0(G) = (f_0)_{G(u,v)}$ and $f(G) = f_{G(u,v)}$, and to determine the β- and γ-duals of these spaces. Furthermore, some classes of matrix mappings on the space $f(G)$ are characterized.

SPACES OF ALMOST NULL AND ALMOST CONVERGENT SEQUENCES

The shift operator P is defined on ω by $(Px)_m = x_{m+1}$ for all $n \in \mathbb{N}$. A Banach limit L is defined on ℓ_∞, as a non-negative linear functional, such that $L(Px) = L(x)$ and $L(e) = 1$. A sequence $x = (x_k) \in \ell_\infty$ is said to be almost convergent to

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the generalized limit α if Banach limits of x is α [8], if Banach limits of x is α, and denoted by $f-\lim\ x_k=\alpha$. Let P^i be the composition of P with itself i times and write for a sequence $x=(x_k)$

$$t_{mn}(x)=\frac{1}{n+1}\sum_{k=0}^{n}(P^i x)_m; \quad (m,n\in\mathbb{N}). \tag{3}$$

Lorentz [8] proved that $f-\lim\ x_k=\alpha$ if and only if $\lim_{n\to\infty}t_{mn}(x)=\alpha$, uniformly in m. It is well-known that a convergent sequence is almost convergent such that its ordinary and generalized limits are equal.

The spaces f_0 and f of almost null and almost convergent sequences are defined, as follows,

$$\begin{aligned} f_0 &= \left\{x=(x_k)\in\omega:\lim_{n\to\infty}t_{mn}(x)=0\ \text{ uniformly in }\ m\right\}, \\ f &= \left\{x=(x_k)\in\omega:\exists\alpha\in\mathbb{C}\ni\lim_{n\to\infty}t_{mn}(x)=\alpha\ \text{ uniformly in }\ m\right\}, \end{aligned}$$

where $t_{mn}(x)$ is defined by (3).

NEW SEQUENCE SPACES AND THEIR DUALS

In this section, we introduce the sequence spaces $f_0(G)$ and $f(G)$, and give some results concerning with them and determine their beta and gamma duals.

Malkowsky and Savaþ [9] have defined the sequence space $Z=(u,v;X)$ which consists of all sequences whose $G(u,v)$-transforms are in $X\in\{\ell_\infty,c,c_0,\ell_p\}$, where $u,v\in U$. The space $Z(u,v;X)$ defined by

$$Z=Z(u,v;X)=\left\{x=(x_j)\in\omega:y=\left(\sum_{j=0}^{k}u_k v_j x_j\right)\in X\right\}.$$

Altay and Baþar [1] have recently constructed new paranormed sequence spaces $\lambda(u,v;p)$ for $\lambda\in\{\ell_\infty,c,c_0\}$ by

$$\lambda(u,v;p)=\left\{x=(x_j)\in\omega:y=\left(\sum_{j=0}^{k}u_k v_j x_j\right)\in\lambda(p)\right\}.$$

The new sequence spaces $f_0(G)$ and $f(G)$ as the set of all sequence whose $G(u,v)$-transforms are in the spaces f_0 and f, that is,

$$\begin{aligned} f_0(G) &= \left\{x=(x_k)\in\omega:\lim_{n\to\infty}t_{mn}(G(u,v)x)=0 \text{ uniformly in } m\right\}, \\ f(G) &= \left\{x=(x_k)\in\omega:\exists\alpha\in\mathbb{C}\ni\lim_{n\to\infty}t_{mn}(G(u,v)x)=\alpha \text{ uniformly in } m\right\}. \end{aligned}$$

By the notation of (2), the sequence spaces $f_0(G)$ and $f(G)$ are restated as

$$f_0(G)=(f_0)_{G(u,v)}\ \text{ and }\ f(G)=f_{G(u,v)}.$$

Theorem 1 *The inclusions $c\subset f(G)$ and $\ell_\infty\subset f(G)$ strictly hold.*

Theorem 2 *The sequence spaces $f(G)$ and $f_0(G)$ are norm isomorphic to the spaces f and f_0, respectively; i.e., $f(G)\cong f$ and $f_0(G)\cong f_0$.*

Theorem 3 *The sequence spaces $f_0(G)$ and $f(G)$ strictly include the spaces f_0 and f, respectively.*

Lemma 1 *[2, Corollary 3.3] The Banach space f has no Schauder basis.*

Since, it is known that the matrix domain λ_A of a normed sequence space λ has a basis if and only if λ has a basis whenever $A=(a_{nk})$ is a triangle (cf. [4, Remark 2.4]) and the space f has no Schauder basis by Lemma 1, we have:

Corollary 1 *The space $f(G)$ has no Schauder basis.*

Theorem 4 *Let $u, v \in U$ and $z = (z_k) \in \omega$. Define the matrix $E = (e_{nk})$ by*

$$e_{nk} = \begin{cases} \frac{1}{u_n}\left(\frac{z_k}{v_k} - \frac{z_{k+1}}{v_{k+1}}\right) & , \quad (k < n) \\ \frac{z_k}{u_k v_k} & , \quad (k = n) \\ 0 & , \quad (k > n) \end{cases}$$

for all $k, n \in \mathbb{N}$. Then,

$$f(G)^{\beta} = \{z = (z_k) \in \omega : E \in (f : c)\}$$

and

$$f(G)^{\gamma} = \{z = (z_k) \in \omega : E \in (f : \ell_{\infty})\}.$$

SOME MATRIX MAPPINGS RELATED TO THE SPACE $f(G)$

In this section, we give two theorems characterizing the classes of matrix transformations from the sequence space $f(G)$ into any given sequence space μ and from any sequence space μ into the given sequence space $f(G)$.

Theorem 5 *Suppose that the entries of the infinite matrices $E = (e_{nk})$ and $F = (f_{nk})$ are connected with the relation*

$$e_{nk} = \sum_{j=k}^{\infty} u_j v_k f_{nj} \quad or \quad f_{nk} = \frac{1}{u_k}\left(\frac{e_{nk}}{v_k} - \frac{e_{n,k+1}}{v_{k+1}}\right) \tag{4}$$

for all $k, n \in \mathbb{N}$ and μ be any given sequence space. Then $E \in (f(G) : \mu)$ if and only if $\{e_{nk}\}_{k \in \mathbb{N}} \in f(G)^{\beta}$ for all $n \in \mathbb{N}$ and $F \in (f : \mu)$.

Theorem 6 *Suppose that the entries of the infinite matrices $A = (a_{nk})$ and $B = (b_{nk})$ are connected with the relation*

$$b_{nk} = \frac{1}{u_k}\left(\frac{a_{nk}}{v_k} - \frac{a_{n,k+1}}{v_{k+1}}\right)$$

for all $k, n \in \mathbb{N}$ and μ be any given sequence space. Then $A \in (\mu : f(G))$ if and only if $B \in (\mu : f)$.

CONCLUSION

As an essential work on the algebraic and topological properties of the spaces f_0 and f, Başar and Kirişçi [2] have recently introduced the sequence spaces $\widehat{f_0}$ and $\widehat{f}$ derived by the domain of generalized difference matrix $B(r,s)$ in the sequence spaces f_0 and f, respectively. Following Başar and Kirişçi [2], Kayaduman and Şengönül have studied the domain $\widetilde{f_0}$ and $\widetilde{f}$ of the Cesàro mean of order one in the spaces f_0 and f, in [5]. They have determined the $\beta-$ and $\gamma-$duals of the new spaces $\widetilde{f_0}$ and $\widetilde{f}$, and characterize some classes of matrix transformations on the new sequence spaces and into the new sequence spaces. They complete the paper by a nice section including some core theorems related to the matrix classes on/in the new sequence space $\widetilde{f}$. Quite recently, in [11], Sönmez has introduced the domain $f(B)$ of the triple band matrix $B(r,s,t)$ in the sequence space f. In this paper, the $\beta-$ and $\gamma-$duals of the spaces $f(B)$ are determined. Furthermore, the classes $(f(B) : \mu)$ and $(\mu : f(B))$ of infinite matrices are characterized together with some other classes, where μ is any given sequence space. Finally, in [3] Candan has studied the sequence spaces $f_0(\widetilde{B})$ and $f(\widetilde{B})$ as the domain of the double sequential band matrix $\widetilde{B}(\widetilde{r},\widetilde{s})$ in the sequence spaces f_0 and f.

Since Kirişçi and Başar [7]; and Başar and Kirişçi [2]; Kayaduman and Şengönül [5]; Sönmez [10, 11] and Candan [3] are recent works on the domain of certain triangle matrices in the spaces f_0, f and in the classical sequence spaces, the present paper is their natural continuation.

Finally, we should note that the investigation of the domain of some particular limitation matrices, namely Cesàro means of order m, Nörlund means, etc., in the spaces f_0 and f will lead us to the new results which are not comparable with the present results.

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